ITP-SB-99-28, SUNY-NTG-99-23

On two-color QCD with baryon chemical potential

J. B. Kogut[a], M. A. Stephanov[b] and D. Toublan[c]

[a] *Loomis Laboratory of Physics, 1110 West Green Street, Urbana, IL 61801-3080, USA*
[b] *Institute for Theoretical Physics, SUNY, Stony Brook, NY 11794-3840, USA*
[c] *Department of Physics, SUNY, Stony Brook, NY 11794-3800, USA*

Abstract

We study SU(2) color QCD with even number of quark flavors. First, using QCD inequalities we show that at finite baryon chemical potential μ, condensation must occur in the channel with scalar diquark quantum numbers. This breaks the U(1) symmetry generated by baryon charge (baryon superconductivity). Then we derive the effective Lagrangian describing low lying meson and baryon excitations using extended *local* chiral symmetry of the theory. This enables us to determine the leading term in the dependence of the masses on μ exactly.

1 Introduction

QCD at finite baryon number density has been intensely studied recently [1, 2, 3, 4, 5]. Knowing the behavior of QCD in this regime will enable us to understand the physics of heavy ion collisions, neutron stars and supernova explosions. First principle calculations using methods of lattice field theory have presented an insurmountable theoretical challenge to date due to the absence of techniques to deal numerically with complex measure path integrals. The two-color QCD model is an exceptional case where conventional methods work due to positivity of the Euclidean path integral measure [6].

The two-color QCD model is also exceptional from the point of view of BCS-type diquark condensation phenomenon which received attention recently [3]. In 3-color QCD such a condensate is not gauge invariant and it leads to the phenomenon of color superconductivity. In 2-color QCD the diquark condensate is a well-defined gauge invariant observable. Before we learn how to deal with the three-color QCD it would be very helpful to get as much insight as possible from the apparently easier (both conceptually and technically) case of two-color QCD.

Numerical calculations in SU(2) QCD are now being actively pursued [7]. In this letter we develop analytical methods which enable us to study two-color QCD at finite baryon number density, and in particular, to determine the spectrum of excitations.

We shall work in the Euclidean formulation of the theory. The Lagrangian is given by:

$$\mathcal{L} = \sum_{f=1}^{N_f} \left[\bar{\psi}_f \gamma_\mu D_\mu \psi_f + \mu \bar{\psi}_f \gamma_0 \psi_f + m_q \bar{\psi}_f \psi_f \right], \tag{1}$$

and we shall omit the flavor indices f in the following. In this letter we shall consider the case of massless quarks, $m_q = 0$. The analysis of the more general massive case will be presented elsewhere. We also illustrate our methods using the simplest case of $N_f = 2$ quark flavors. The results can be easily extended to arbitrary even N_f. It is known that the 2-color $N_f = 2$ theory with massless quarks at $\mu = 0$ possesses SU(4) global flavor symmetry which is broken spontaneously to Sp(4) [8]. As a result the spectrum contains 5 Goldstone bosons. At finite μ the symmetry of the theory is reduced to the usual SU(2)×SU(2)×U(1). We shall show, using QCD inequalities and, independently, the exact effective Lagrangian, that this symmetry is spontaneously broken down to SU(2)×SU(2), creating a single Goldstone boson corresponding to spontaneous breaking of baryon number symmetry. The other 4 Goldstones acquire a common mass which is proportional to μ for small $\mu \ll \Lambda_{\text{QCD}}$.[1] We find that the coefficient of proportionality can be determined *exactly*, and is equal to 2.

2 QCD inequalities

In Euclidean QCD, having a positive measure, one can majorate all correlators with the correlator $\langle \pi(x)\pi(0) \rangle$, where $\pi = \bar{u}\gamma_5 d$ is the pion field [9]. Therefore, one can prove that

[1] Such a linear dependence on μ can also be seen in the simple effective sigma-model of Rapp et al. [3]. Note that this linear dependence of Goldstone masses on μ contrasts with the usual dependence on another symmetry breaking parameter, the quark mass: $m_\pi \sim \sqrt{m_q}$.

0^- is the lightest meson with $I = 1$. As a consequence, one obtains an important restriction on the pattern of the symmetry breaking: it has to be driven by a condensate $\langle\bar{\psi}\psi\rangle$ (not $\langle\bar{\psi}\gamma_5\psi\rangle$, for example, which would give 0^+ Goldstones).

Let us sketch the argument. Consider the Dirac operator in QCD: $\mathcal{D} = \gamma\cdot(\partial + A) + \mu\gamma_0 + m_q$. When $\mu = 0$ this operator obeys (matrix A is antihermitian in Euclidean formulation, while the γ-matrices are hermitian):

$$\gamma_5\mathcal{D}\gamma_5 = \mathcal{D}^\dagger. \tag{2}$$

Now consider the correlator of a generic meson: $M = \bar{\psi}\Gamma\psi$:

$$\langle M(x)M(0)\rangle = \langle \mathrm{Tr}\mathcal{S}(x,0)\Gamma\mathcal{S}(0,x)\Gamma\rangle, \tag{3}$$

where we did the obvious integration over the ψ's and $\bar{\psi}$'s and left the integration over the A's (it is important, as is true for $I = 1$, that there is no disconnected piece). $\mathcal{S} \equiv \mathcal{D}^{-1}$. When $\Gamma = \gamma_5$ we can use (2) to rewrite the expression in brackets as:

$$\mathrm{Tr}\mathcal{S}(x,0)\mathcal{S}^\dagger(x,0) \tag{4}$$

which is manifestly positive (the dagger in this formula only transposes the color and Dirac indices, the coordinate indices $x, 0$) we transposed explicitly). Moreover, for any Γ (such that $\Gamma^2 = 1$) we can write, using the Schwartz inequality:

$$\mathrm{Tr}\mathcal{S}(x,0)\Gamma\mathcal{S}(0,x)\Gamma = \mathrm{Tr}\mathcal{S}(x,0)\Gamma\gamma_5\mathcal{S}^\dagger(x,0)\gamma_5\Gamma \le \mathrm{Tr}\mathcal{S}(x,0)\mathcal{S}^\dagger(x,0). \tag{5}$$

If the measure is positive this inequality should survive the averaging, and we get the desired inequality for the correlators, and therefore for the meson masses.[2]

For $\mu \neq 0$ we lose the positivity and we lose the inequalities in SU(3). But, in SU(2) QCD we also have a positive measure! Can we, perhaps, derive some inequalities for the meson masses and consequently make some conclusions about the symmetry breaking pattern?

The relation (2) holds in either SU(3) or SU(2). It also fails in both theories at $\mu \neq 0$. But there is another relation, which holds in SU(2), due to its pseudo-reality, for *arbitrary* μ:

$$\gamma_5 C T_2 \mathcal{D}\gamma_5 C T_2 = \mathcal{D}^*, \tag{6}$$

where $C = i\gamma_0\gamma_2$ ($C^2 = 1$, $C\gamma_\mu C = -\gamma_\mu^*$) all γ-matrices are hermitian, and T_2 is a generator of the SU(2) color (the second Pauli matrix, $T_2 T_a T_2 = -T_a^*$). It is a consequence of this relation that the measure is positive, in fact.

If we construct now the correlator of the diquark $M_{\psi\psi} = \psi^T C T_2 \gamma^5 \psi$ (this is 0^+, $I = 0$, i.e., antisymmetric in flavor), we have:

$$\langle M_{\psi\psi}(x) M_{\psi\psi}^\dagger(0)\rangle = \langle \mathrm{Tr}\mathcal{S}(x,0) C T_2 \gamma^5 \mathcal{S}^T(x,0) C T_2 \gamma^5\rangle = \langle \mathrm{Tr}\mathcal{S}(x,0)\mathcal{S}^\dagger(x,0)\rangle. \tag{7}$$

Now, as before, one can show that the correlator of $\psi^T C T_2 \gamma^5 \psi$ meson majorates a correlator of any other meson $\psi^T C T_2 \gamma^5 \Gamma \psi$. In particular, we see that it is 0^+, not 0^-, which is the

[2] To make this argument into a mathematical theorem one would have to consider regularization. We shall not pursue this level of rigor here and shall refer the interested reader to original literature [9].

lightest. Therefore, if there is condensation it has to be that of $\psi^T C T_2 \gamma^5 \psi$, not violating parity, in particular.

One can also majorate the correlator of any meson of the type $\bar{\psi}\Gamma\psi$. In other words, the 0^+ diquark must be the lightest meson in this case. This excludes the possibility of conventional condensation of $\langle\bar{\psi}\psi\rangle$ which otherwise would lead to 3 massless pions (unless the inequality is saturated, which is the case at $\mu = 0$).

3 Symmetries, breaking and Goldstone counting

We shall construct the effective Lagrangian describing light excitations in 2-color QCD at finite μ in the next section. Here we shall analyze the global symmetries of our theory — a necessary ingredient of this construction.

We start from the known case of $\mu = 0$ and recall the fact that the global symmetry of the theory is $SU(2N_f)$ rather than the usual $SU(N_f)\times SU(N_f)\times U(1)$ [8]. This can be seen explicitly by using left and right chiral Weyl components of the Dirac spinor $\psi = (q_L, q_R)$:

$$\mathcal{L} = \bar{\psi}\gamma_\mu D_\mu \psi = q_L^\dagger i\sigma_\mu D_\mu q_L + q_R^\dagger i\bar{\sigma}_\mu D_\mu q_R, \tag{8}$$

where $\sigma_\mu = (-i, \sigma_k)$ and $\bar{\sigma}_\mu = (-i, -\sigma_k)$, and σ_k are usual Pauli matrices. The fact that (1) has higher flavor symmetry is based on the property of the 2-color Dirac operator, $D_\mu = \partial_\mu + A_\mu$ (A is antihermitian SU(2) color generator matrix $A = A^a T_a$, where T_a are color generators):

$$D_\mu^T = -T_2 D_\mu T_2 \tag{9}$$

which in turn is based on the (pseudoreality) property of the generators of SU(2) (Pauli matrices):

$$T_a^* = (T_a^T =) - T_2 T_a T_2. \tag{10}$$

We introduce:

$$\tilde{q} = \sigma_2 T_2 q_R^\dagger, \quad \text{and} \quad \tilde{q}^\dagger = q_R^T T_2 \sigma_2. \tag{11}$$

We then substitute (11) into (8) and use the property (10) of Pauli matrices for both T_a of color and σ_k of Euclid, together with the anticommutativity of $\tilde{q}$, $\tilde{q}^\dagger$ (we need to transpose) to arrive at:

$$\mathcal{L} = q^\dagger i\sigma_\mu D_\mu q + \tilde{q}^\dagger i\sigma_\mu D_\mu \tilde{q} = \Psi^\dagger i\sigma_\mu D_\mu \Psi. \tag{12}$$

which now has a manifest $SU(2N_f)$ “flavor” symmetry. The Ψ denotes a Weyl spinor which has $2N_f$ “flavor” components. E.g., for $N_f = 2$:

$$\Psi = \left(\begin{array}{c} q \\ \tilde{q} \end{array}\right) = \left(\begin{array}{c} q^1 \\ q^2 \\ \tilde{q}^1 \\ \tilde{q}^2 \end{array}\right), \tag{13}$$

where $1, 2$ are the original flavor indices.

The total global symmetry of the action is $SU(2N_f)\times U(1)_A$. Note that the baryon symmetry, under which $B(q) = +1$ and $B(\tilde{q}) = -1$ is a subgroup of this $SU(2N_f)$. The $\tilde{q}$

are, therefore, conjugate quarks (since they have opposite baryon charge to normal quarks q) in the terminology of [2]. Under axial $\mathrm{U}(1)_A$ q and $\tilde{q}$ have the same charge (because $A(\tilde{q}) = -A(q_R) = A(q_L) = A(q)$). This symmetry is broken by the anomaly, however, so the actual symmetry of the quantum field theory is $\mathrm{SU}(2N_f)$.

Now, let us write down various useful quark bilinears in terms of q, $\tilde{q}$ and determine their transformation properties under this $\mathrm{SU}(2N_f)$.

$$\bar{\psi}\psi = q_R^\dagger q_L + q_L^\dagger q_R = \tilde{q}^T \sigma_2 T_2 q + q^\dagger \sigma_2 T_2 (\tilde{q}^\dagger)^T = \frac{1}{2}\Psi^T \sigma_2 T_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Psi + \text{h.c.} \; . \tag{14}$$

The matrix in (14) is a $2N_f \times 2N_f$ matrix in the $\mathrm{SU}(2N_f)$ indices. The matrices σ_2 and T_2 carry SU(2) spin and SU(2) color indices respectively and no $\mathrm{SU}(2N_f)$ indices. They are just antisymmetric ϵ-symbols for their indices. We see that the chiral condensate is not a singlet under $\mathrm{SU}(2N_f)$. Since it is an antisymmetric product of two fundamental $\mathrm{SU}(2N_f)$ spinors Ψ, it transforms as an antisymmetric tensor of rank 2. The dimension of this representation is $N_f(2N_f - 1)$.

We shall continue our discussion using $N_f = 2$ case as an example. For $N_f = 2$ (14) transforms as a 6-plet. The (14) gives us one component of this 6-plet (sigma). The remaining 5 are: 3 pions, scalar diquark and anti-diquark.

What does the chemical potential do?

$$\bar{\psi}\gamma_0\psi = q_L^\dagger q_L + q_R^\dagger q_R = q^\dagger q + \tilde{q}^T (\tilde{q}^\dagger)^T = q^\dagger q - \tilde{q}^\dagger \tilde{q} = \Psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi. \tag{15}$$

We see that this term is not a singlet under SU(4). Since $4 \times 4 = 1 + 15$ it is a component of a 15-plet (adjoint representation $(2N_f)^2 - 1$). It is easy to understand the meaning of $+1$ and -1 in the matrix in (15) — these are just baryon charges of quarks and conjugate quarks.

What is the remaining subgroup of SU(4), under which (15) is invariant? From the block-diagonal structure of (15) it is clear that $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ rotations preserve it, since these rotate the first two components of Ψ, or the last two, separately. The $\mathrm{U}(1)_B$, which can be thought as generated by the block τ_3 generator (the charges are $(+,+,-,-)$), also preserves (15). All other generators are broken by (15). To summarize, we start with an SU(4) symmetry; then we add a term proportional to μ which transforms as a component of a 15-plet of this SU(4) which breaks this symmetry *explicitly* down to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B$.

Now let us do the Goldstone counting. At $\mu = 0$ we have SU(4) global symmetry. The non-zero expectation value of the quark bilinear (14) which develops spontaneously breaks it down to Sp(4). This produces 5 Goldstone bosons (15 generators minus 10). On the other hand, when $\mu \neq 0$ the symmetry of the theory is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B$. As we concluded in the previous section this symmetry should break down to $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ by the non-zero expectation value of scalar diquark. Therefore at $\mu \neq 0$ the theory has only one Goldstone. What happened to the other 4? As is easy to guess, and as we shall see explicitly, they form a representation $(\underline{2}, \underline{2})$ of the manifest $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ group, and acquire the same mass. This mass should vanish at $\mu = 0$. In the next section we shall calculate the dependence of the mass of the 4-plet of these pseudo-Goldstones, m_{pG} as a function of μ for small μ.

4 Effective Lagrangian

4.1 Global symmetry

In this section we construct the effective Lagrangian for the low energy degrees of freedom, which in our theory with spontaneous symmetry breaking are the Goldstone bosons [10]. The basic steps we follow are: (i) identify the symmetries of the underlying (microscopic) theory; (ii) identify degrees of freedom of the effective (macroscopic) theory; (iii) ensure that the effective theory is invariant under the symmetries of the microscopic theory.The microscopic theory at $\mu = 0$ has a global SU(4) symmetry. In the effective theory, which we want to construct, the degrees of freedom are given by the fluctuations of the condensate of Σ:

$$\Sigma \sim \Psi\Psi^T \sigma_2 T_2, \tag{16}$$

which is a Lorentz and color singlet but flavor SU(4) 6-plet. Fluctuations of the orientation of Σ give us our 5 Goldstones. Under the action of $U \in$SU(4):

$$\Psi \to U\Psi \tag{17}$$

and thus

$$\Sigma \to U\Sigma U^T. \tag{18}$$

The low-energy effective Lagrangian invariant under the flavor SU(4) can be written as a non-linear sigma model [10, 8]:

$$\mathcal{L}_1 = f_\pi^2 \mathrm{Tr}\partial_\mu \Sigma^\dagger \partial_\mu \Sigma \ . \tag{19}$$

The matrix Σ in the effective Lagrangian is a unitary antisymmetric matrix (which has exactly 5 independent real parameters). The degrees of freedom are the rotations of Σ generated by U as in (18). The transformations U which leave Σ invariant form the Sp(4) group. The nontrivial degrees of freedom of the Lagrangian (19) – the Goldstones – live in the coset SU(4)/Sp(4).

In the microscopic theory, the term:

$$\mu\bar{\psi}\gamma_0\psi = \mu\Psi^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi. \tag{20}$$

breaks the SU(4) symmetry explicitly. However, we can save this symmetry by transforming also the source coupled to the breaking term. Rewriting (20) as

$$\mu\Psi^\dagger i\sigma_\mu B_\mu \Psi, \tag{21}$$

where B_μ is an SU(4) matrix. The value of B_μ fixed by (20) is:

$$B_\mu = \delta_{0\mu} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{22}$$

If under the transformation (17) the matrix B_μ also transforms as:

$$B_\mu \to U B_\mu U^\dagger, \tag{23}$$

the microscopic Lagrangian

$$\Psi^\dagger i\sigma_\mu D_\mu \Psi + \mu \Psi^\dagger i\sigma_\mu B_\mu \Psi \tag{24}$$

will be invariant. Thus the effective Lagrangian must be also invariant under such an extended transformation. For example, this requirement rules out the term linear in B (and therefore in μ) in the effective Lagrangian. This is because B transforms as (23) under SU(4), and one cannot construct a non-trivial invariant out of Σ and only one power of B. The lowest order nontrivial term which we can write is:

$$\mu^2 \mathrm{Tr} \Sigma B^T \Sigma^\dagger B. \tag{25}$$

This term will produce the mass for the Goldstone bosons *linear* in μ.

4.2 Local symmetry

We see that the symmetry considerations help us find the form of the symmetry breaking term in the effective Lagrangian, and thus determine the dependence of the m_{pG} on μ. However, it does not tell us what the coefficient of proportionality in $m_{\mathrm{pG}} \sim \mu$ is, since it does not specify the coupling of (25). This coefficient can be determined if we notice that the global symmetry (17), (23) can in fact be promoted to a *local* symmetry in the microscopic theory [10]. This will require the transformation of B:

$$B_\mu \to U B_\mu U^\dagger + \frac{1}{\mu} U \partial_\mu U^\dagger. \tag{26}$$

In order to ensure that the effective Lagrangian is also invariant under this local symmetry we have to replace the derivatives in (19) by the long covariant derivatives:

$$\begin{aligned} D_\mu \Sigma &= \partial_\mu \Sigma + \mu (B_\mu \Sigma + \Sigma B_\mu^T) \\ D_\mu \Sigma^\dagger &= \partial_\mu \Sigma^\dagger - \mu (\Sigma^\dagger B_\mu + B_\mu^T \Sigma^\dagger). \end{aligned} \tag{27}$$

The signs here are important and are fixed by the local symmetry. The Lagrangian must have the form:

$$\mathcal{L}_{\mathrm{eff}} = f_\pi^2 \mathrm{Tr} D_\mu \Sigma^\dagger D_\mu \Sigma \tag{28}$$

Expanding the long derivatives we find:

$$\mathcal{L}_{\mathrm{eff}} = f_\pi^2 \left[\mathrm{Tr} \partial_\mu \Sigma^\dagger \partial_\mu \Sigma - 4\mu \mathrm{Tr} \partial_\mu \Sigma \Sigma^\dagger B_\mu - \mu^2 \mathrm{Tr} (\Sigma^\dagger B_\mu + B_\mu^T \Sigma^\dagger)(B_\mu \Sigma + \Sigma B_\mu^T) \right] , \tag{29}$$

where we used the property $\Sigma^T = -\Sigma$ to simplify the second term. Now we shall analyze the last term in (29). There are two main effects of this term. First, the minimum of it with respect to all possible orientations of Σ obtained by rotations (18) determines the direction of the condensation. Second, the curvature matrix around this minimum gives us the masses for the (pseudo)-Goldstones.

We see that the local symmetry relates the mass term to the kinetic term in the Lagrangian. This means that the coefficient of proportionality in the equation $m_{\mathrm{pG}} = \mathrm{const} \cdot \mu$, which is just a *dimensionless* number, is fixed by the local chiral symmetry! In particular, f_π does not enter at all into this relation. In the remainder of this note we shall set $f_\pi = 1$ to simplify the formulas.

4.3 Vacuum alignment

Using the fact that $B_\mu^\dagger = B_\mu$, we can see that the last term in (29) is seminegative definite:

$$\mathcal{L}_3 = -\mu^2 \mathrm{Tr} A A^\dagger, \tag{30}$$

where

$$A = B\Sigma + \Sigma B^T. \tag{31}$$

In order to find the vacuum alignment of Σ we must minimize $\mathcal{L}_3$. Let us try first the alignment corresponding to the usual chiral condensate:

$$\Sigma = \Sigma_{\bar{\psi}\psi} \equiv \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{32}$$

Using B given by (22) we find that $A = 0$ and therefore $\mathcal{L}_3 = 0$ which is the absolute maximum of $\mathcal{L}_3$, not the minimum which we seek. Therefore the standard vacuum alignment (with no baryon charge in the condensate) is unstable.

One can see that the minimum can be achieved for a $\Sigma = \Sigma_0$ such that:

$$B\Sigma_0 = \Sigma_0 B^T. \tag{33}$$

A solution to (33) is given by:

$$\Sigma_0 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \tag{34}$$

This minimum is not unique, there is a U(1) degeneracy, corresponding to the rotation with the generator given by B_0 (22). This gives the Goldstone corresponding to the spontaneous breaking of the baryon charge symmetry. Any other rotation will raise the value of the effective potential.

4.4 Mass spectrum

Now we shall consider the curvature of the potential in more detail, to determine m_{pG}. We can rewrite (30) as:

$$\mathcal{L}_3 = -2\mu^2 \mathrm{Tr} B_\mu^2 - 2\mu^2 \mathrm{Tr} \Sigma B_\mu^T \Sigma^\dagger B_\mu. \tag{35}$$

The dependence on Σ sits in the second term. In order to find the mass matrix for the (pseudo)-Goldstones we should expand Σ in small fluctuations around the vacuum value Σ_0 (34). These small fluctuations are given in terms of the transformation (18) with U close to unity:

$$\Sigma = U\Sigma_0 U^T. \tag{36}$$

We shall write U as an exponent of the generators of the SU(4). But first let us, following the formalism and notations of Peskin [11], separate the generators into those which do not change Σ_0 — T_i, and those that do — X_a. The transformations U generated by T_i:

$$U = e^{i\phi_i T_i} \qquad \text{such that} \qquad U\Sigma_0 U^T = \Sigma_0, \tag{37}$$

form an Sp(4) subgroup of SU(4). It follows from (37) that these generators obey:

$$T_i \Sigma_0 = -\Sigma_0 T_i^T. \tag{38}$$

The remaining 5 generators X_a can be shown, using the block representation of Peskin, to obey:

$$X_a \Sigma_0 = +\Sigma_0 X_a^T. \tag{39}$$

The corresponding fields π_a defined as:

$$U = e^{i\pi_a X_a}, \tag{40}$$

and by (36) are the dynamical degrees of freedom of the Lagrangian (29).

We shall write here, to provide an example, the explicit form of the generators T and X in our case of the SU(4) flavor group. There are 10 generators T_a:

$$T_{1-3} = \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \ T_{4-6} = \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix}, \ T_{7-9} = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \ T_{10} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{41}$$

And here are the 5 generators X_a:

$$X_{1-3} = \begin{pmatrix} 0 & i\sigma_i \\ -i\sigma_i & 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{42}$$

We have normalized the generators as:

$$\mathrm{Tr} T_i T_k = \delta_{ik} \mathrm{Tr} 1, \quad \mathrm{Tr} X_a X_b = \delta_{ab} \mathrm{Tr} 1, \quad \mathrm{Tr} X_a T_i = 0, \tag{43}$$

where $\mathrm{Tr}1 = 4$.

Let us make the following observations. First, B should be one of the generators of SU(4). From (33) and (39) we conclude that it has to belong to the set of broken generators X_a. Our explicit example confirms this, indeed $X_5 = B$. Second, the remaining generators ($a = 1,2,3,4$ in our example) anticommute with B.

For a transformation U generated by X which *commutes* with B we can write for the last term in (35):

$$-2\mathrm{Tr}\Sigma B_\mu^T \Sigma^\dagger B_\mu = -2\mathrm{Tr} U \Sigma_0 U^T B_\mu^T U^* \Sigma_0^\dagger U^\dagger B_\mu = -2\mathrm{Tr} B_\mu^2, \tag{44}$$

which is a constant. So there is no mass for the corresponding boson. We conclude that π_5 is a true Goldstone.

For the remaining 4 generators X we can write, using the fact that they anticommute with B:

$$\begin{aligned} -2\mathrm{Tr}\Sigma B_\mu^T \Sigma^\dagger B_\mu &= -2\mathrm{Tr} U \Sigma_0 U^T B_\mu^T U^* \Sigma_0^\dagger U^\dagger B_\mu \\ &= -2\mathrm{Tr} U^2 \Sigma_0 (U^T)^2 B_\mu^T \Sigma_0^\dagger B_\mu = -2\mathrm{Tr} U^4, \end{aligned} \tag{45}$$

where we have used properties of the X generators (39), (33) and $B^2 = 1$. Now using (40), (43) and expanding to quadratic order in the fields we find:

$$-2\mathrm{Tr} U^4 = +16\pi_a \pi_b \mathrm{Tr} X_a X_b + \mathcal{O}(\pi^4) = 16\pi_a^2 \mathrm{Tr} 1 + \mathcal{O}(\pi^4), \tag{46}$$

where $a = 1-4$. Now expanding the kinetic term we find, using (36) and (39):

$$\mathrm{Tr}\partial_\mu\Sigma^\dagger\partial_\mu\Sigma = \mathrm{Tr}\partial_\mu(U^{-2})\partial_\mu(U^2) = 4\partial_\mu\pi_a\partial_\mu\pi_b\mathrm{Tr}X_aX_b + \mathcal{O}(\pi^4) = 4(\partial_\mu\pi_a)^2\mathrm{Tr}1 + \mathcal{O}(\pi^4), \quad (47)$$

where $a = 1-5$. Desired normalization of the kinetic term can be trivially achieved by rescaling the fields π_a. Taking together (46), (47) and (29) we find:

$$m_{\mathrm{pG}} = m_{1-4} = 2\mu \qquad \text{and} \qquad m_5 = 0. \quad (48)$$

This is our result.

4.5 Linear term

What is the significance of the second term in (29):

$$-4\mu\mathrm{Tr}\partial_\mu\Sigma\Sigma^\dagger B_\mu \ ? \quad (49)$$

Let us expand it for the generators X using (43):

$$-4\mu\mathrm{Tr}\partial_\mu\Sigma\Sigma^\dagger B_\mu = -8i\mu\partial_\mu\pi_5\mathrm{Tr}X_5B_\mu + \mathcal{O}(\pi_5^3). \quad (50)$$

Only the generator X_5 gives a nonvanishing contribution to the linear term in π, and the fact that $\mathrm{Tr}X_5X_5B = 0$ ensures that there are no terms quadratic in π_5. This linear term means that B_μ (the baryon charge current) is a source of the Goldstone field π_5 (similar to the axial current in QCD being the pion source).

5 Generic even N_f

Most of the derivation goes through *mutatis mutandis* in the general case. Here we shall summarize the results. The global symmetry at $\mu = 0$ is $\mathrm{SU}(2N_f)$. This symmetry is broken spontaneously:

$$\mathrm{SU}(2N_f) \xrightarrow{\Sigma} \mathrm{Sp}(2N_f), \qquad (\mu = 0). \quad (51)$$

The number of the Goldstones in this case is:

$$((2N_f)^2 - 1) - \frac{1}{2}\,2N_f(2N_f+1) = 2N_f^2 - N_f - 1, \qquad (\mu = 0). \quad (52)$$

At nonzero μ the symmetry is broken down to:

$$\mathrm{SU}(2N_f) \xrightarrow{\mu} \mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1). \quad (53)$$

The condensate, being an antisymmetric rank 2 tensor (cf. (34), breaks it now in the following way:

$$\mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1) \xrightarrow{\Sigma} \mathrm{Sp}(N_f) \times \mathrm{Sp}(N_f). \quad (54)$$

Note that in the case $N_f = 2$: Sp(2)$\sim$ SU(2), so only U(1) is broken. When $N_f > 2$ we shall have more than one true Goldstone boson. Their number is:

$$2(N_f^2-1) + 1 - 2\frac{1}{2}N_f(N_f+1) = N_f^2 - N_f - 1 \qquad (\mu \neq 0). \quad (55)$$

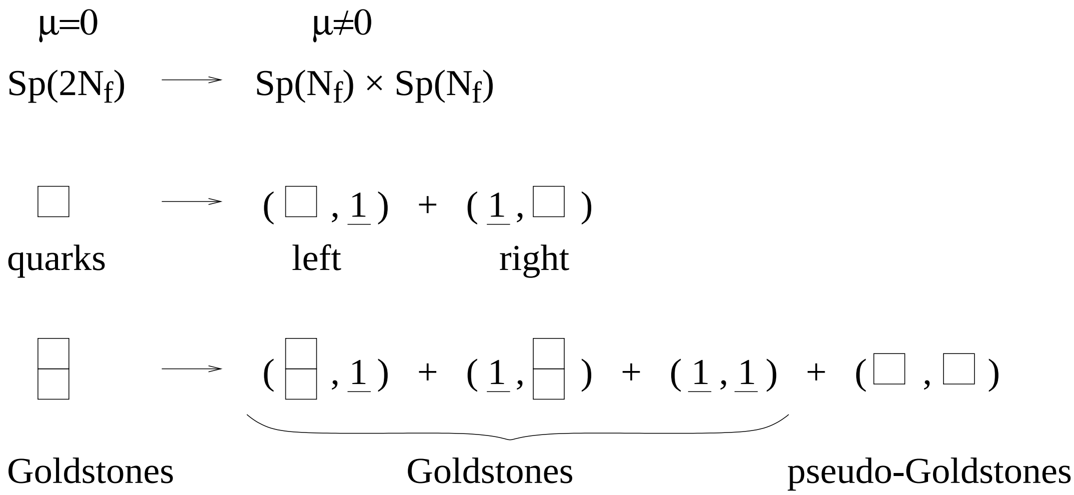

Figure 1: Young's diagrams illustrating the breakdown of the Goldstone fields into multiplets at $\mu = 0$ and at $\mu \neq 0$. The quarks, which transform in the fundamental representation, are given first as an example.

Comparing (52) and (55) we find that there are N_f^2 pseudo-Goldstone bosons. Their masses are given by (48): $m_{\rm pG} = 2\mu$ for small μ.

In terms of group representations, we have the following picture. First, $\mu = 0$. The fermions transform as a fundamental $2N_f$-plet under $\mathrm{SU}(2N_f)$. The fermion condensate transforms as an antisymmetric tensor of rank 2. The dimension of this representation is $N_f(2N_f - 1)$. After the breaking to $\mathrm{Sp}(2N_f)$ the Goldstones fall into an irreducible representation of $\mathrm{Sp}(2N_f)$ given by the antisymmetric tensor of rank 2 with the condition that trace of that tensor times the matrix Σ_0 is zero. The dimension of this representation is $N_f(2N_f - 1) - 1$ which is exactly (52).

The baryon charge current to which μ couples transforms in the adjoint representation of $\mathrm{SU}(2N_f)$, which has dimension $(2N_f)^2 - 1$.

After the spontaneous breakdown (54) the N_f^2 pseudo-Goldstones are degenerate and form an $(\underline{N_f}, \underline{N_f})$ irreducible representation of the remaining manifest $\mathrm{Sp}(N_f)\times\mathrm{Sp}(N_f)$. The true Goldstones fall into 3 irreducible representations: a singlet $(\underline{1}, \underline{1})$, $(\underline{N_f(N_f - 1)/2 - 1}, 1)$ and $(1, \underline{N_f(N_f - 1)/2 - 1})$, with the total count given by (55). The irreducible representation $\underline{N_f(N_f - 1)/2 - 1}$ of $\mathrm{Sp}(N_f)$ is the antisymmetric tensor of rank 2 with the condition that the trace of that tensor times a certain antisymmetric matrix vanishes (this representation does not exist for $N_f = 2$). This breakdown is convenient to view in terms of Young diagrams in Fig. 1.

Understanding the multiplet structure turns out to be very important in the analysis of the spectrum at small μ *and* small quark mass m_q. This analysis will be presented elsewhere.

6 Conclusions

In this letter we used two methods to study the physics of 2-color QCD at finite baryon chemical potential. We used the fact that the measure of the Euclidean path integral in such a theory remains positive definite even at finite μ to derive certain inequalities between non-singlet meson correlators. These inequalities translate into inequalities between masses of the lightest mesons and impose strong restrictions on possible patterns of the symmetry breaking. In particular, we show that the lightest meson is the 0^+ diquark, and therefore condensation (if it occurs) must occur in the channel $\psi^T C\gamma_5\psi$ thus leading to baryon charge superconductivity. This fact is in perfect agreement with model calculations which show that both instanton-induced and one-gluon exchange interactions are most attractive in this channel [3].

We also derived the low-energy effective Lagrangian describing the mesons and baryons of the 2-color QCD with *massless* quarks. We found that both the sign and the magnitude of the coefficient of the potential term of this Lagrangian is fixed by a *local* chiral symmetry. The sign determines the pattern of the spontaneous symmetry breaking, and is such that it agrees with the QCD inequalities. The masses of the mesons as a function of μ can be also determined exactly for small μ. For example, in the case of $N_f = 2$ flavors of quarks the low energy spectrum at small μ consists of one massless particle, and a 4-plet of massive particles with masses equal to 2μ.

This result can be understood physically. The massless particle is the Goldstone of the broken symmetry generated by the baryon charge. It has the quantum numbers of a scalar diquark $\psi^T C\gamma_5\psi$. The SU(2)×SU(2) 4-plet of massive mesons is comprised of the sigma ($\bar\psi\psi$) and 3 pions ($\bar\psi\gamma_5\tau_i\psi$). These excitations could be thought of as loosely bound pairs, or threshold states of a quark and an antiquark similar to the sigma particle in the Nambu-Jona-Lasinio model [12] with mass $2m_{\mathrm{fermion}}$. In the rest frame, the quark is taken from the surface of the Fermi sea, with momentum $|\boldsymbol{p}| = \mu$, while the antiquark has the opposite momentum $-\boldsymbol{p}$, thus making up the invariant mass of 2μ. One should be aware, however, of the fact that such a description is intuitive at best, since the ground state can be described by the Fermi sphere only in the absence of interactions between quarks. In the theory we consider the quarks are confined.

We would like to thank S. Nussinov, E. Shuryak, J. Verbaarschot and A. Zhitnitsky for very fruitful and stimulating discussions. Many of the issues discussed here have been addressed in the context of Lattice Gauge Theory. J.B.K. thanks M.-P. Lombardo, S. Hands, S. Morrison and D.K. Sinclair for discussions. J.B.K. is supported in part by the National Science Foundation, NSF-PHY96-05199. M.S. is supported by the grant NSF-PHY97-22101. D.T. is partially supported by the US DOE grant DE-FG-88ER40388 and by "Holderbank"-Stiftung.

References

[1] J. B. Kogut, M. P. Lombardo and D. K. Sinclair, Phys. Rev. D 51 (1995) 1282; Nucl. Phys. B, Proc. Suppl. 42 (1995) 514; I.M. Barbour, S.E. Morrison, E.G. Klepfish, J.B. Kogut, M.P. Lombardo, Nucl. Phys. Proc. Suppl. **60A** (1998) 220.

[2] M.A. Stephanov, Phys. Rev. Lett. **76** (1996) 4472; Nucl. Phys. Proc. Suppl. **53** (1997) 469.

[3] D. Bailin and A. Love, Phys. Rept. **107** (1984) 325; R. Rapp, T. Schäfer, E.V. Shuryak and M. Velkovsky, Phys. Rev. Lett. **81** 1998 53; M. Alford, K. Rajagopal and F. Wilczek, Phys. Lett. **B422** (1998) 247, Nucl. Phys. **B537** (1999) 215; J. Berges and K. Rajagopal, Nucl. Phys. **B538** (1999) 215; T. Schäfer and F. Wilczek, Phys. Rev. Lett. 82 (1999) 3956; hep-ph/9903503; M. Alford, J. Berges and K. Rajagopal, hep-ph/9903502; R.D. Pisarski and D.H. Rischke, nucl-th/9811104; D.T. Son, Phys. Rev. **D59** (1999) 094019; E. Shuster and D.T. Son, hep-ph/9905448.

[4] M.A. Halasz, A.D. Jackson, R.E. Shrock, M.A. Stephanov and J.J.M. Verbaarschot, Phys. Rev. **D58** (1998) 096007.

[5] M. Stephanov, K. Rajagopal, and E. Shuryak, Phys. Rev. Lett. **81** (1998) 4816.

[6] E. Dagotto, F. Karsch, and A. Moreo, Phys. Lett. B **169** (1986) 421; E. Dagotto, A. Moreo, and U. Wolff, Phys. Rev. Lett. **57** (1986) 1292; Phys. Lett. B **186** (1987) 395.

[7] S. Hands, J.B. Kogut, M.-P. Lombardo, S.E. Morrison, hep-lat/9902034; S. Hands and S.E. Morrison, hep-lat/9902012, hep-lat/990521.

[8] A. Smilga and J.J.M. Verbaarschot, Phys. Rev. **D51** (1995) 829.

[9] D. Weingarten, Phys. Rev. Lett. **51** (1983) 1830; E. Witten, Phys. Rev. Lett. **51** (1983) 2351; S. Nussinov, Phys. Rev. Lett. **52** (1984) 966; D. Espriu, M. Gross and J.F. Wheater, Phys. Lett. B **146** (1984) 67.

[10] J. Gasser and H. Leutwyler, Ann. Phys. **158**, 142 (1984); Nucl. Phys. **B250**, 465 (1985); G. Ecker, Prog. Part. Nucl. Phys. **35** (1995) 1; D. Toublan and J.J.M. Verbaarschot, hep-th/9904199.

[11] M.E. Peskin, Nucl. Phys. **B175** (1980) 197.

[12] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122** (1961) 345.